1 Numerical sets

Monoid

Let A be a non-empty set and $\bullet : A \times A \to A$ be a binary operation. The pair $(A, \bullet)$ is said *monoid* if the following holds:

i) for every $a, b, c \in A$ we have

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \qquad \textit{(associative property)} \tag{1.1}$$

ii) there exists an element $e \in A$ such that $\forall a \in A$ we have

$$a \bullet e = e \bullet a = a \qquad \textit{(neutral element)} \tag{1.2}$$

Additionally, if the following holds

iii) $\forall a, b \in A$

$$a \bullet b = b \bullet a \qquad \textit{(commutative property)} \tag{1.3}$$

then the pair $(A, \bullet)$ is said to be commutative (or Abelian) monoid.

Example, Natural numbers

Consider the set of natural numbers

$$\mathbb{N} := \{0, 1, 2, 3, \dots\} \tag{1.4}$$

Let $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ the usual sum and multiplication. Then, $(\mathbb{N}, +)$ is a monoid with neutral element $e_+ = 0$ and the pair $(\mathbb{N}, \cdot)$ is a monoid with neutral element $e_\cdot = 1$.

Group

Let A be a non-empty set and $\bullet : A \times A \to A$ be a binary operation. The pair $(A, \bullet)$ is said *group* if the following holds:

i) for every $a, b, c \in A$ we have

$$(a \bullet b) \bullet c = a \bullet (b \bullet c) \qquad \textit{(associative property)} \tag{1.5}$$

ii) there exists an element $e \in A$ such that $\forall a \in A$ we have

$$a \bullet e = e \bullet a = a \qquad \textit{(neutral element)} \tag{1.6}$$

iv) $\forall a \in A$ there exists $a^{-1} \in A$ such that

$$a \bullet a^{-1} = a^{-1} \bullet a = e \qquad \textit{(inverse)} \tag{1.7}$$

Additionally, if the following holds

iii) $\forall a, b \in A$

$$a \bullet b = b \bullet a \qquad \textit{(commutative property)} \tag{1.8}$$

then the pair $(A, \bullet)$ is said to be commutative (or Abelian) group.

Ring

Let A be a non-empty set and $\bullet : A \times A \to A$ and $\star : A \times A \to A$ be two binary operations. The triple $(A, \bullet, \star)$ is said *ring* if $(A, \bullet)$ is an Abelian group, if $(A, \star)$ is a monoid and if the following holds (*distributive property*):

v) $\forall a, b, c \in A$ we have

$$a \star (b \bullet c) = (a \star b) \bullet (a \star c) \qquad \textit{and} \qquad (b \bullet c) \star a = (b \star a) \bullet (c \star a) \tag{1.9}$$

Example, Integer numbers

Consider the set of the integer numbers

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}. \tag{1.10}$$

Let $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $\cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ the usual sum and multiplication. Then the triple $(\mathbb{Z}, +, \cdot)$ is a ring with neutral element $e_+ = 0$ for the sum and $e_\cdot = 1$ for the product.

Field

Let A be a non-empty set and $\bullet : A \times A \to A$ and $\star : A \times A \to A$ be two binary operations. The triple $(A, \bullet, \star)$ is said *field* if $(A, \bullet)$ is an Abelian group with neutral element $e_\bullet \in A$, if $(A^\star, \star)$ is an Abelian group with neutral element $e_\star \in A$ and if (1.9) holds for every $a, b, c \in A$ where $A^\star := A \backslash e_\bullet$.

Example, Rational numbers

Consider the set of rationals

$$\mathbb{Q} := \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}^\star\} \tag{1.11}$$

where $\mathbb{Z}^\star = \mathbb{Z} \backslash \{0\}$. Let $+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$, $\cdot : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ the standard sum and product. Then, the triple $(\mathbb{Q}, +, \cdot)$ is a field with neutral element $e_+ = 0$ for the sum and $e_\cdot = 1$ for the product. Observe $\mathbb{Q}^\star = \mathbb{Q} \backslash \{0\}$.

Real numbers

The system of real numbers is the quadruple $(\mathbb{R}, +, \cdot, \leq)$ where

1. $(\mathbb{R}, +, \cdot)$ is a field;
2. $(\mathbb{R}, \leq)$ is a totally ordered set;
3. every set $\emptyset \neq I \subseteq \mathbb{R}$ bounded from above admits supremum ($\sup I$) (analogously, every set $\emptyset \neq I \subseteq \mathbb{R}$ bounded from below admits infimum ($\inf I$).