LECTURE 6, SEPTEMBER 17

BASIC/FUNDAMENTAL PROPERTIES OF FUNCTIONS

1) DEFINITION   MONOTONIC FUNCTIONS

$f: \mathbb{R} \to \mathbb{R}$ $(f, \mathbb{R}, \mathbb{R})$

$f: A \to B$ (f, A, B) $A, B \subseteq \mathbb{R}$

LET $A, B \subseteq \mathbb{R}$ $f: A \to B$

WE SAY THAT f IS

- MONOTONICALLY INCREASING IF

  $x_1 < x_2$ $f(x_1) < f(x_2)$

  $\forall x_1, x_2 \in A$;

- MONOTONICALLY NON-DECREASING IF

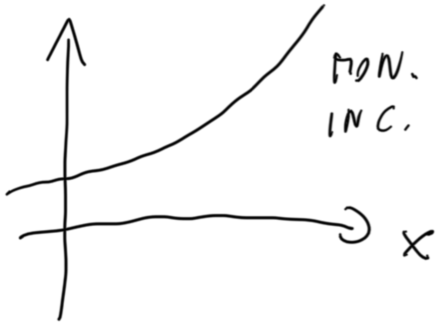
  $\forall x_1, x_2 \in A$ $x_1 < x_2$ $f(x_1) \leq f(x_2)$

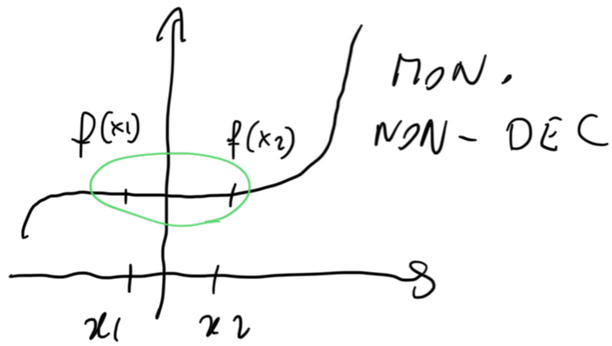
- MONOTONICALLY DECREASING IF

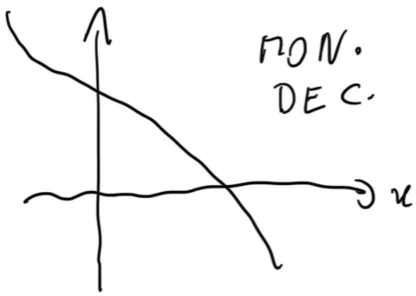
$\forall x_1, x_2 \in A \quad x_1 < x_2 \qquad f(x_1) > f(x_2)$

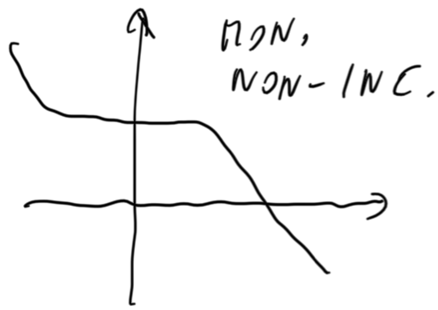
- MONOTONICALLY NON-INCREASING

$\forall x_1, x_2 \in A \quad x_1 < x_2 \qquad f(x_1) \geq f(x_2)$









- DEFINITION 2) $A \subseteq \mathbb{R}$ A IS SAID "SYMMETRIC" IF $x \in A \Rightarrow$ ALSO $-x \in A$

$\mathbb{R}$ IS SYMMETRIC.

- DEFINITION 3) $f: A \to \mathbb{R}$ IS SAID

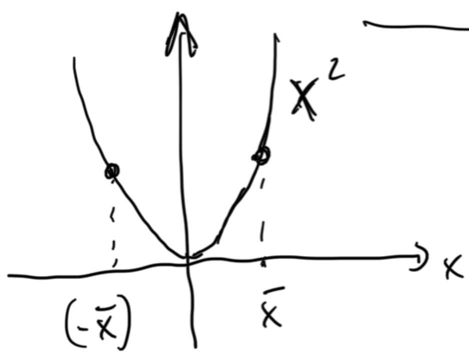
"EVEN" IF $f(x) = f(-x) \quad \forall x \in A$

"ODD" IF $f(x) = -f(-x) \quad \forall x \in A$

Ex. $f(x) = x^2$ $\quad ((\cdot)^2, \mathbb{R}, \mathbb{R})$

IT'S EVEN $\quad f(x) = f(-x)$

$$x^2 = (-x)^2 \quad \forall x \in \mathbb{R}$$

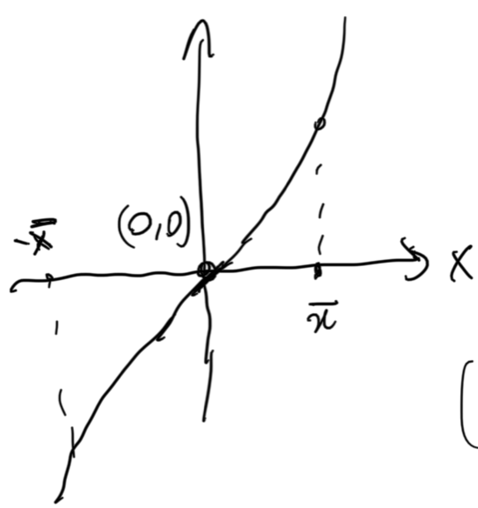


→ SEE EXERCISE SHEET # 2

$f(x) = x^3$ ODD $\quad f(x) \stackrel{?}{=} -f(-x)$

$$x^3 \stackrel{?}{=} -(-x)^3$$

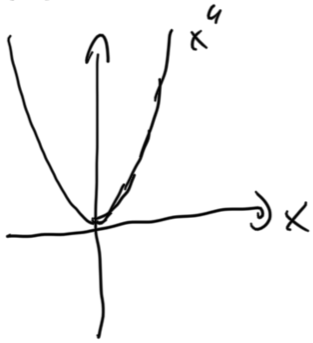
$$+x^3$$



$$(\bar{x})^3 = -(-\bar{x})^3$$

$f(x) = x^4$ $\quad f : \mathbb{R} \to \mathbb{R}$

EVEN



$$x^4 = (-x)^4 \quad \forall x \in \mathbb{R}$$

$\boxed{n \in \mathbb{N} \setminus \{0\}}$

"EVEN" $f(x) := x^n$ IT'S EVEN IF n IS EVEN

"ODD" IT'S ODD " n " ODD

<u>REMARK</u>

IN GENERAL EVEN FUNCTIONS ARE NOT INJECTIVE

IN GENERAL ODD FUNCTIONS ARE INJECTIVE

<u>REMARK</u> $f(x) = K$ K IS A CONSTANT $K \in \mathbb{R}$

IF $K \neq 0$ $f(x) = K$ $(K, \mathbb{R}, \mathbb{R})$
IS EVEN

IF $K = 0$ IS BOTH EVEN + ODD

<u>PROPOSITION</u> $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

1) IF f & g ARE EVEN $\Rightarrow$ $fg : \mathbb{R} \to \mathbb{R}$ IS EVEN

2) IF f & g ARE ODD $\Rightarrow$ " "

3) IF f IS EVEN g IS ODD
f IS ODD g IS EVEN $\Rightarrow$ $fg : \mathbb{R} \to \mathbb{R}$ IS ODD

PROOF $\to$ VERIFY THE DEFINITION – CONSIDER 2)

$f(x) = -f(-x)$ $\qquad$ $g(x) = -g(-x)$

$fg : \mathbb{R} \to \mathbb{R}$ $\qquad$ $(fg)(x) \overset{(*)}{=} f(x)\, g(x)$

$$(fg)(x) \overset{?}{=} (fg)(-x)$$

$$f(x)\, g(x) = f(-x)\, g(-x)$$
$$[-f(x)]\ [-g(x)]$$

TRY YOURSELF OTHER CASES

$\boxed{\text{POWERS}}$ $\qquad$ x^n $\qquad$ $n \in \mathbb{N} \setminus \{0\}$

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f : (\cdot)^n$

$x^2 \to$ WE HAVE STUDIED THIS FUNCTION ALREADY LAST WEEK

$(x^2, \mathbb{R}, \mathbb{R})$ IS NOT INJECTIVE

$(x^2, \mathbb{R}, \mathbb{R}^+)$ IS SURJ. NOT INJ.

$(x^n, \mathbb{R}, \mathbb{R}^+)$ $\boxed{n = \text{EVEN}}$ IS SURJ. NOT INJECTIVE

INSTEAD WHEN n IS EVEN

$(x^n, \mathbb{R}^+, \mathbb{R}^+)$ IS BI-JECTIVE

AND WE CAN COMPUTE THE INVERSE FUNCTION $f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$

$f := \cdot^2 ; \mathbb{R}^+ \to \mathbb{R}^+$ $\qquad f^{-1} := \sqrt{\cdot} ; \mathbb{R}^+ \to \mathbb{R}^+$

x^4 $\qquad \sqrt[4]{\cdot} ; \mathbb{R}^+ \to \mathbb{R}^+$

x^n $\qquad\qquad$ $\sqrt[n]{\cdot} : \mathbb{R}^+ \to \mathbb{R}^+$

$\boxed{n \text{ ODD}}$

$f(x) = (\cdot)^1 : \mathbb{R} \to \mathbb{R}$

$(\cdot)^3$

$\boxed{n \text{ IS ODD}}$

$\left((\cdot)^n, \mathbb{R}, \mathbb{R}\right)$ ARE ALL SURJECTIVE
INJECTIVE
BI-JECTIVE

$f := (\cdot)^n : \mathbb{R} \to \mathbb{R}$ $\qquad$ $f^{-1} := \sqrt[n]{\cdot} : \mathbb{R} \to \mathbb{R}$

A FEW MORE PROPERTIES

ON THE INTERVAL $(0,1)$ $\quad x^n > x^m$ IF $n < m$

$(+1, \infty)$ $\quad x^n > x^m$ IF $n > m$

$(0,1)$ $\quad \sqrt[n]{x} > \sqrt[m]{x}$ IF $n > m$

$(1, \infty)$ $\quad \sqrt[n]{x} > \sqrt[m]{x}$ IF $n < m$

EXPONENTIALS AND LOGARITHMS

$a \in \mathbb{R}$, $a > 0$, $a \neq 1$ ⊛

$f(x) = a^x$ $f: \mathbb{R} \to \mathbb{R}$

FOR VARIOUS VALUES OF a

THESE ARE EXAMPLES OF MONOTONIC FUNCTIONS

IF $a > 1 \to a^x$ IS MON. INCREASING

OR $a < 1 \to a^x$ IS MON. DECREASING.

$(a^x, \mathbb{R}, \mathbb{R}^+ \setminus \{0\})$ a AS ABOVE ⊛

IS BIJECTIVE

THANKS TO (STRICT) MONOTONICITY

$a > 1$ FOR $x_1, x_2 \in \mathbb{R}$ $x_1 < x_2$

MON. INCR.

$a^{x_1} < a^{x_2}$

REMEMBER INJECTIVITY; $x_1 \neq x_2 \quad f(x_1) \neq f(x_2)$

$a < 1$ $\qquad x_1, x_2 \in \mathbb{R} \quad x_1 < x_2$

MON. DECR.

$$a^{x_1} > a^{x_2}$$

$\rightarrow$ INJECTIVE

$$\mathbb{R}^+ := \{ x \in \mathbb{R} \ ; \ x \geq 0 \}$$

INVERSE FUNCTION IS LOGARITHM

$$f^{-1} ; \ \mathbb{R}^+ \setminus \{0\} \rightarrow \mathbb{R}$$

$$\log_a(\cdot)$$

"THE EXPONENTIAL" FUNCTION

e^x $\qquad e \in \mathbb{R} \quad e$ "NEPER" NUMBER

$\simeq 2,71828...$

WE WILL DEFINE THE NEPER'S NUMBER LATER

$$e > 1$$

$\log_e \triangleq \ln$

$\log$

$\log_a(\cdot) : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}$ MON. INC. $a > 1$

$\mathbb{R}^+ \setminus \{0\} \to \mathbb{R}$ MON. DEC. $a < 1$

PROPERTY - EXP, log NOT EVEN NOR ODD

DEFINITION "EVEN PART" OF A FUNCTION

"ODD PART"

$f : \mathbb{R} \to \mathbb{R}$

$$f_{EVEN}(x) := \frac{f(x) + f(-x)}{2} \quad (**) \qquad f_{ODD}(x) := \frac{f(x) - f(-x)}{2}$$

FIRST PROPERTY $f(x) = f_{EVEN}(x) + f_{ODD}(x)$

PROOF
$$\left[\frac{f(x) + \cancel{f(-x)}}{2} + \frac{f(x) - \cancel{f(-x)}}{2} \right] = \underline{f(x)}$$

f IS "DECOMPOSED" INTO f_{EVEN} AND f_{ODD}

AND f_{EVEN}, f_{ODD} ARE UNIQUELY IDENTIFIED BY (**)

PROPERTY f_{EVEN} IS EVEN

f_{ODD} IS ODD

PROOF f_{EVEN} IS EVEN. $f_{EVEN}(x) = f_{EVEN}(-x)$

$$\frac{f(x) + f(-x)}{2} = \frac{f(-x) + f(-(-x))}{2}$$

AS REQUIRED □

PROVE YOURSELF f_{ODD} IS ODD

EXAMPLE $f_{EVEN}(x^2)$ $\quad$ $f_{ODD}(x^2)$

$$\frac{x^2 + (-x)^2}{2} = \frac{x^2 + x^2}{2} = x^2$$

$$f_{ODD}(x^2) = \frac{x^2 - (-x)^2}{2} = 0$$

<u>PROPERTY</u> EVEN ODD PART OF AN EVEN ODD FUNCTION COINCIDES WITH THE FUNCTION ITSELF WHILE THE ODD EVEN PART OF AN EVEN ODD FUNCTION IS 0

---

sinh, cosh

HYPERBOLIC TRIGONOMETRIC FUNCTIONS

$$\cosh(x) = := \frac{e^x + e^{-x}}{2} \qquad \checkmark$$

$\cosh(\cdot) : \mathbb{R} \to \mathbb{R}$

$$\sinh(x) := (e^x - e^{-x})/2 \qquad \checkmark$$

$\sinh : \mathbb{R} \to \mathbb{R}$

"HYPERBOLIC COSINUS"

" SINUS

" TANGENT "

$$\tanh(x) ; \frac{\sinh(x)}{\cosh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = +1$$

VERIFY YOURSELF...

$\cosh ; \mathbb{R} \to [1, \infty)$

SURJECTIVE
NOT INJECTIVE
(EVEN)

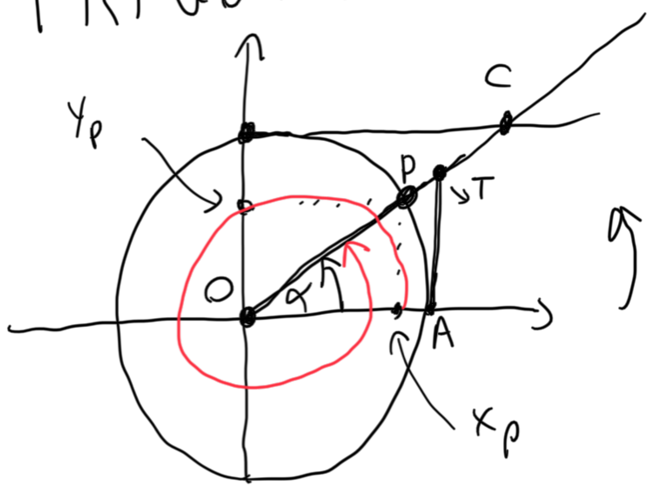
$\sinh ; \mathbb{R} \to \mathbb{R}$

SURJECTIVE
INJECTIVE
(ODD)
MONOTONICALLY INCREASING
BIJECTIVE

$\tanh ; \mathbb{R} \to (-1, 1)$

SURJECTIVE
INJECTIVE
(ODD)
MONOTONICALLY INCREASING
BIJECTIVE

TRIGONOMETRY



UNIT CIRCLE $\overline{PO} = 1$

α

α IN RADIANS

$$\sin \alpha = y_p$$

$$\cos \alpha = x_p$$

$$\tan \alpha = y_T = \frac{\sin \alpha}{\cos \alpha}$$

$$\cotan \alpha = x_C = \frac{\cos \alpha}{\sin \alpha}$$

$x \in \mathbb{R}$

$\sin(x)$ $\quad$ $\cos(x)$

$(\sin(\cdot), \mathbb{R}, \mathbb{R})$ $\quad$ $(\cos(\cdot), \mathbb{R}, \mathbb{R})$

$\sin^2(x) + \cos^2(x) = 1 \quad \forall x \in \mathbb{R}$

DEFINITION_ $f: \mathbb{R} \to \mathbb{R}$ AND A CONSTANT NUMBER $T \in \mathbb{R}$; $T > 0$

WE SAY THAT f IS PERIODIC WITH PERIOD EQUAL TO T (OR f IS T-PERIODIC)

IF $f(x) = f(x + T) \quad \forall x \in \mathbb{R}$ (***)

WE DEFINE T_{FUND} THE FUNDAMENTAL PERIOD, THE SMALLEST T, THAT IS

$$T_{FUND} := \inf \left\{ \underline{T > 0} ; f(x+T) = f(x) \ \forall x \in \mathbb{R} \right\}$$

<u>EXAMPLE</u> $(1, \mathbb{R}, \mathbb{R})$

$$1 \equiv f[x+T) = f(x) \equiv 1 \quad \forall x \in \mathbb{R}$$

ANY POSITIVE NUMBER T WILL DO

$T = 1 \quad T = 100 \quad T = 5$

THE CONSTANT FUNCTION IS PERIODIC WITH ANY PERIOD

BUT IT DOES NOT HAVE FUNDADENTAL PERIOD

TRIGONOM. FUNCTIONS sin, cos, ARE PERIODIC FUNCTIONS

$(\sin(\cdot), \mathbb{R}, [-1,1])$ IS SURJ.

$(\cos(\cdot), \mathbb{R}, [-1,1])$ IS SURJ.

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \quad \text{PROVIDED } \cos(x) \neq 0$$

$$\tan, \mathbb{R} \setminus \left\{ (2k+1)\frac{\pi}{2}, k \in \mathbb{Z} \right\}$$

$\uparrow$

POINTS MAKING COSINUS ZERO.