LECTURE 5 SEPTEMBER 15

$I = \{ r \in \mathbb{R} \quad 0 \leq r < \sqrt{2} \}$

I IS BOUNDED FROM BELOW AND FROM ABOVE

$\sqrt{2}$ IS THE SUPREMUM OF I

$\sqrt{2} \notin \mathbb{Q}$ AS WE SAW LAST LECTURE

$\sqrt{2} \in \mathbb{R}$

$I = \{ r \in \mathbb{Q} \quad 0 \leq r < \sqrt{2} \}$

DOES NOT HAVE A SUPREMUM IN $\mathbb{Q}$!!

WITH THE AXIOMATIC DEFINITION OF $\mathbb{R}$

WE "COMPLETE" $\mathbb{Q}$

"INTERVALS" IN $\mathbb{R}$

$(a, b) := \{ x \in \mathbb{R} \; ; \; a < x < b \}$

$$[a, b] := \{x \in \mathbb{R} ; a \leq x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} ; a \leq x < b\}$$

$$(a, b] := \{x \in \mathbb{R} ; a < x \leq b\}$$

$$a, b \in \mathbb{R}$$

HALF-LINES – WE ASSUME $a = -\infty$

$$(-\infty, b) := \{x \in \mathbb{R} ; x < b\}$$

$$(-\infty, b] := \{x \in \mathbb{R} ; x \leq b\}$$

WE ASSUME $b = +\infty$

$$(a, \infty) := \{x \in \mathbb{R} ; x > a\}$$

$$[a, \infty) := \{x \in \mathbb{R} ; x \geq a\}$$

NEIGHBORHOOD

FIX $x_0 \in \mathbb{R}$, WE SAY THAT $U = (a, b)$

$a, b \in \mathbb{R}$ (POSSIBLY $a = -\infty$, $b = +\infty$) $a < b$

IS A NEIGHBORHOOD OF x_0 IF $x_0 \in U$

SYMMETRIC NEIGHBORHOODS

FIX $x_0 \in \mathbb{R}$, FIX $r > 0$, THE SET

$$\underline{I_r(x_0) := (x_0 - r, x_0 + r)} = \{x \in \mathbb{R} : x_0 - r < x < x_0 + r\}$$

IS CALLED A SYMMETRIC NEIGHBORHOOD OF x_0

RIGHT (LEFT) NEIGHBORHOOD

FIX $x_0 \in \mathbb{R}$, $r > 0$

$$I_r^+(x_0) := (x_0, x_0 + r) = \{x \in \mathbb{R} ; 0 < x - x_0 < r\}$$

IS CALLED THE RIGHT NEIGH. OF x_0

$$I_r^-(x_0) := (x_0 - r, x_0) = \{x \in \mathbb{R} ; 0 < x - x_0 < r\}$$

<u>REMARK</u>

$$\underbrace{I_r^+(x_0) \cup I_r^-(x_0)}_{\text{"PUNCTURED" NEIGH.}} = I_r(x_0) \setminus \{x_0\}$$

PROPOSITION

TAKE $x_0 \in \mathbb{R}$, THEN INTERVALS OF THE TYPE

1) $U = (a, b)$, WITH $a < x_0 < b$ AND $a, b \in \mathbb{R}$ SUCH THAT $x_0 \in U$

2) FIX $r > 0$, $I_r(x_0)$ DEFINED ABOVE

ARE "EQUIVALENT". THIS MEANS THAT GIVEN AN INTERVAL U OF THE TYPE 1) THERE EXISTS A NEIGH. OF x_0 OF THE TYPE 2) S.T. $I_r(x_0) \subseteq U$

AND GIVEN A NEIGH. $I_r(x_0)$ THERE EXISTS AN INTERVAL OF THE TYPE 1) S.T. $U \subseteq I_r(x_0)$.

PF 1) GIVEN $U = (a, b)$ $\quad (x_0 - a) > 0$

$(b - x_0) > 0$

$$\min\{(x_0-a), (b-x_0)\} := r$$

$I_r(x_0) \subseteq U$ FIRST IMPLICATION IS PROVED

II) IMPLICATION. I AM FIXING $r > 0$

$I_r(x_0)$. IT'S EVEN EASIER TO VERIFY THAT I CAN CONSTRUCT ANOTHER GENERIC INTERVAL $U \subseteq I_r(x_0)$

$\square$

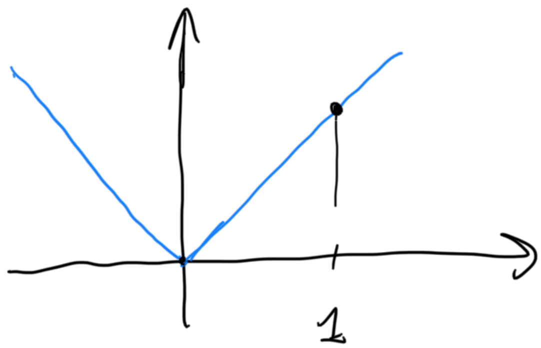
WHAT WE WANT TO SHOW TODAY IS "DENSITY" OF RATIONAL NUMBERS IN THE REAL NUMBERS.

ABSOLUTE VALUE

$(|\cdot|, \mathbb{R}, \mathbb{R})$ $\quad |\cdot| : \mathbb{R} \to \mathbb{R}$ IN THE FOLLOWING WAY:

$$|x| = \begin{cases} x & \text{IF } x \geq 0 \\ -x & \text{IF } x < 0 \end{cases} \quad (*)$$

$$\begin{cases} x & \text{IF } x > 0 \\ -x & \text{IF } x \leq 0 \end{cases}$$



$|1| = 1 = |-1|$

$|0| = 0$

$\boxed{\text{PROPOSITION } 1}$ $\forall Q \in \mathbb{R}$, $Q > 0$.

$\forall x \in \mathbb{R}$ $\quad |x| \leq Q$ COINCIDES WITH

$$-Q \leq x \leq Q$$

PROOF 1) TAKE $|x| \leq Q$ AND USE $(*)$

IF $x \geq 0$ $\quad |x| = x \quad \Rightarrow \quad -Q \leq x \leq Q$

AS REQUIRED.

$x < 0 \quad |x| = -x \quad -x \leq Q$

$$\underset{\text{POSITIVE}}{Q} \geq \underset{\text{NEGATIVE}}{x} \geq -Q$$

AS REQUIRED.

2) CONVERSE IMPLICATION IS EVEN EASIER!
TRY AS AN EXERCISE.

CLASSES THIS WEEK:

- FRIDAY 17 SEPTEMBER 6TH
- SATURDAY 18 SEPTEMBER 7TH
- ~~MONDAY 20 SEPTEMBER~~ CANCELED

INEQUALITIES

1) $\forall x, y \in \mathbb{R}$ "TRIANGULAR INEQ."
$$|x+y| \leq |x| + |y|$$

2) $$|xy| = |x|\,|y|$$

3) $$\big|\,|x| - |y|\,\big| \leq |x-y|$$
"REVERSE TR. INEQ."

PROOF OF 1). WE USE PROPOSITION 1).

$$|x| = |x| \le |x|$$

$$-|x| \le x \le |x|$$ BY APPLYING PROPOSITION 1

REPEAT FOR Y $\quad -|y| \le y \le |y|$

$$-(|x|+|y|) \le x+y \le |x|+|y|$$

BUT THIS IS EQUIVALENT to: $|x+y| \le |x|+|y|$

$\rightarrow$ THIS PROVES 1)

2) WE HAVE TO VERIFY ALL POSSIBLE CASES

$x \ge 0$ & $y \ge 0$ $\qquad x = |x| \qquad y = |y|$

$xy \ge 0$ $\qquad xy = |xy| \qquad \underbrace{|xy|}_{= xy} = \underbrace{|x||y|}_{= xy}$

$x \ge 0$ & $y < 0$ $\qquad x = |x| \qquad y = -|y|$

$xy \le 0$ $\qquad xy = -|xy|$

$$|xy| = -xy$$

WE SHOW $|xy| \overset{?}{=} |x|\,|y| = (x)\underline{(-y)}$

$$\begin{array}{c} \| \\ (-xy) \end{array}$$

<u>REMARK FROM 2)</u> $|x| = |-x|$

IT IS ENOUGH TO TAKE $y = -1$

$|y| = |-1| = +1$

3) $\underline{|x|} = |x - y + y| = |(x-y) + y| \leq$

APPLY 1) TRIANGULAR INEQ.

$\leq \underbrace{|x-y| + |y|}$

$|x| - |y| \leq |x-y|$ $(**)$

$|y| = |y-x+x| \leq |y-x| + |x|$ BY 1)

$$\begin{array}{c} |y-x| \\ \| \\ |x-y| \end{array}$$

$|y| - |x| \leq |x-y|$

MULTIPLY BOTH SIDES BY -1

$|x| - |y| \geq -|x-y|$ $(***)$

NOW APPLY PROP. 1 TO (*)(*) & (*)(*)(*)

AND YOU HAVE $\Rightarrow$ $\quad a = |x-y|$

$$\big|\, |x| - |y| \,\big| \leq |x-y|$$

$\square$

PROPOSITION 2 $\forall x \in \mathbb{R} \quad \exists n \in \mathbb{N}:$

$$x < n \quad (****).$$

PROOF BY CONTRADICTION, THE "ABSURDUM" ARGUMENT IS (****) IS FALSE.

$$\exists \bar{x} \in \mathbb{R} : \bar{x} \geq n \quad \forall n \in \mathbb{N}$$

THIS MEANS $\bar{x}$ IS AN UPPER BOUND FOR THE SET OF THE NATURAL NUMBERS IN $\mathbb{R}$

BY AXIOMATIC DEFINITION OF $\mathbb{R}$

$\Rightarrow \exists\, a \in \mathbb{R}$ THE SUPREMUM OF THE NATURAL NUMBERS IN $\mathbb{R}$

$\Rightarrow \underline{\forall n \in \mathbb{N}} \quad n \leq Q$

$n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N} \Rightarrow n+1 \leq Q$

BUT Q IS THE SUPREMUM $\quad n \leq Q-1$

$Q-1$ IS AN UPPER BOUND TO THE NATURAL NUMBERS

BUT THIS IS A CONTRADICTION BECAUSE
THE SUP THE SMALLEST OF THE UPPER BOUNDS $\square$

PROPOSITION 3 $\quad \forall x \in \mathbb{R} \quad \exists n \in \mathbb{Z}:$

$$n \leq x < n+1 \quad \text{(SANDWICH)}$$

PROOF $\rightarrow$ SIMILAR TO PROOF OF PROP. 2

DEFINITION THE INTEGER n FOUND IN
PROP. 3 IS CALLED "INTEGER PART"
OF x $\quad [x] :=$ INTEGER PART OF x

$\sqrt{2} \quad n=1 \quad 1 \leq \sqrt{2} < 2$

$\sqrt{2} = 1.416\ldots \quad [\sqrt{2}] = 1$

$\pi \in \mathbb{R}$ $\quad \pi = 3.14159\ldots$ $\quad [\pi] = 3$

$3 \leq \pi < 4 \quad \ldots$

$\boxed{\begin{array}{l} \underline{\text{PROPOSITION 4}} \quad \text{"DENSITY" OF } \mathbb{Q} \text{ IN } \mathbb{R} \\ \forall x \in \mathbb{R} \text{ AND } \forall \varepsilon > 0 \quad \exists r \in \mathbb{Q} : \\ r \leq x < r + \varepsilon \end{array}}$

PROOF TAKE $x \in \mathbb{R}$, FIX $\varepsilon > 0$

$\frac{1}{\varepsilon} \in \mathbb{R}$ APPLY PROP. 2 TO $\frac{1}{\varepsilon}$

$q \in \mathbb{N}$ $\quad \frac{1}{\varepsilon} < q \quad q \neq 0$

$$\varepsilon > \frac{1}{q}$$

NOW I APPLY PROP. 3 TO $qx \in \mathbb{R}$

$$p \leq qx < p+1 \qquad p := [qx] \in \mathbb{Z}$$

DIVIDE BY q $(\neq 0)$ AND OBTAIN:

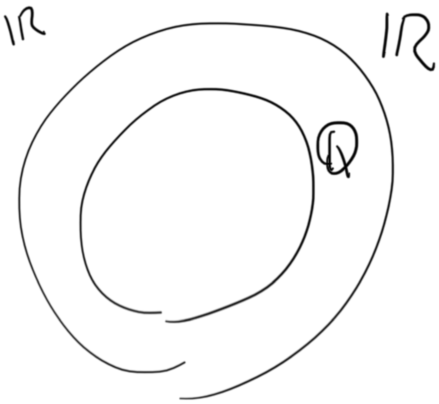
$$\frac{p}{q} \leq x < \frac{p}{q} + \frac{1}{q} = \frac{p}{q} + \varepsilon$$

$$p \in \mathbb{Z} \quad q \in \mathbb{N} \setminus \{0\} \Rightarrow p/q \in \mathbb{Q}$$

$$\frac{p}{q} := r$$

$$r \leq x < r + \varepsilon \quad \text{AS REQUIRED} \qquad \square$$

$\mathbb{Q} \subseteq \mathbb{R}$



$\mathbb{Q}$ IS A DENSE SUBSET OF $\mathbb{R}$ IN THE SENSE OF PROP. 4.