LECTURE 4, SEPTEMBER 13

PROPOSITION

$f : A \to B$ BIJECTIVE

$g : B \to C$ BIJECTIVE , THEN:

- $(g \circ f)^{-1} : C \to A$ IS BIJECTIVE

AND

- $(g \circ f)^{-1}(c) = a = f^{-1} \circ g^{-1}(c)$

$\forall c \in C$ SUCH THAT $c = g(f(a))$

PROOF

$$(\underbrace{g \circ f}_{h})^{-1} \circ (\underbrace{g \circ f}_{h}) = Id_A$$ BY REMARK

$Id_A : A \to A$

$\forall a \in A$

$$a = Id_A(a) = (g \circ f)^{-1} \circ (g \circ f)(a) =$$

(1) $$\underbrace{(g \circ f)^{-1}(c)}_{=}$$

$$\underbrace{(g \circ f)(a) = g(f(a)) = c}$$

$$c = (g \circ f)(a) = g(f(a))$$

NOW APPLY g^{-1} BOTH SIDES

$$g^{-1}(c) = \underbrace{g^{-1} \circ g}_{Id_B}(f(a)) = f(a)$$

$g: B \to C$

$g^{-1}: C \to B$

$g^{-1} \circ g : Id_B$

NOW APPLY f^{-1} BOTH SIDES

$$f^{-1} \circ g^{-1}(c) = \underbrace{f^{-1} \circ f}_{Id_A}(a) = a$$

(2)

$f: A \to B$

$f^{-1}: B \to A$

LET'S PUT ① & ② TOGETHER

$$(g \circ f)^{-1}(c) = a = f^{-1} \circ g^{-1}(c)$$

$\forall\, a \in A$ SUCH THAT $a = g(f(a))$ $\square$

→ MAXIMUN AND MINIMUM

A IS A NON-EMPTY SET

$(A, \leq)$ TOTALLY ORDERED

· WE SAY THAT $\bar{m} \in A$ IS THE MAXIMUM

IF $a \leq \bar{m} \quad \forall\, a \in A$

IF $\bar{m}$ EXISTS WE DENOTE IT WITH $\bar{m} = \max A$

· WE SAY THAT $\underline{m} \in A$ IS THE MINIMUM

IF $\underline{m} \leq a \quad \forall\, a \in A$

IF $\underline{m}$ EXISTS WE DENOTE IT WITH $\underline{m} = \min A$

EXAMPLE 1) $A = \{1, 2, 10, 20\}$

$\# A = 4$ $\quad \overline{m} = \max A = 20$

$\underline{m} = \min A = 1$

2) $A = \{ r \in \mathbb{Q} : 0 \leq r \leq 10 \}$

$\overline{m} = \max A = 10$

$r \leq 10 \quad \forall r \in A$ BY DEFINITION OF A

$\underline{m} = 0 \quad 0 \leq r \quad \forall r \in A$

3) $A = \{ r \in \mathbb{Q} : 0 \leq r < 10 \}$

$\overline{m} \overset{?}{=} 10 \notin A$

$9 \; ? \; \leq 9.9 \leq 9.99$

THERE EXISTS NO MAXIMUM IN A

B.T.W. 0 IS STILL MINIMUM

4) $A = \{ r \in \mathbb{Q} ; 0 \leq r^2 \leq 2 \}$

$= \{ r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2} \}$

WHAT IS THE MAXIMUM HERE?

$m \stackrel{?}{=} \sqrt{2} \quad \forall r \in \mathbb{Q} \quad r \leq \sqrt{2}$

$\sqrt{2} \notin A$ BECAUSE IS NOT A RATIONAL NUMBER !!

BY THE WAY, $\sqrt{2}$ IS NOT A RATIONAL NUMBER, IN FACT $\sqrt{2}$ IS A REAL NUMBER. WHAT DOES IT MEAN? IT IS NOT POSSIBLE TO WRITE

$\sqrt{2} = p/q \qquad p, q \in \mathbb{Z} \qquad q \neq 0$

PROOF THAT $\sqrt{2} \notin \mathbb{Q}$

BY CONTRADICTION, WE ASSUME (ABSURDUM ARGUMENT) THAT $\sqrt{2} = \dfrac{m}{n}$. $\quad n \neq 0$

$n, m \in \mathbb{N}$

M, N HAVE NO COMMON DIVISORS

$\frac{33}{3} = \frac{11}{1}$

(THEY ARE ALREADY REDUCED TO IRREDUCIBLE FORM)

$2 = \frac{m^2}{n^2} \Rightarrow m^2 = 2n^2 \Rightarrow m^2$ IS EVEN

$\Rightarrow m$ IS EVEN

$\Rightarrow \exists k \in \mathbb{N} : m = 2k$

$2n^2 = 4k^2 \Rightarrow n^2 = 2k^2$

$\Rightarrow n^2$ IS EVEN $\Rightarrow n$ IS EVEN

$\Rightarrow$ BUT THIS IS NOT POSSIBLE !

IF BOTH m, n ARE EVEN $\rightarrow$

THEY DO HAVE COMMON DIVISORS! $\rightarrow$

CONTRADICTION $\square$.

UNIQUENESS PROPERTY OF THE MAXIMUM AND THE MINIMUM

IF MAXIMUM (MINIMUM) EXISTS, MUST BE !

PROOF BY CONTRADICTION, SUPPOSE $\exists$ TWO MAXIMUMS $m_1, m_2 \in A$. S.T. $m_1 \neq m_2$

1) $\forall a \in A \quad a \leq m_1 \quad , m_1 \in A$

2) $\forall a \in A \quad a \leq m_2 \quad , m_2 \in A$

$$\left.\begin{array}{ll} 1) \Rightarrow & m_2 \leq m_1 \\ 2) & m_1 \leq m_2 \end{array}\right\} \quad m_1 = m_2$$

$\square$

SAME PROOF FOR MINIMUM

" UPPER BOUND, LOWER BOUND "

DEFINITION. LET $M \neq \emptyset$ AND $(M, \leq)$ TOTALLY ORDERED. WE TAKE $\emptyset \neq A \subseteq M$. WE SAY THAT $m \in M$ IS AN UPPER BOUND FOR A

IF $a \leq m \quad \forall a \in A$

WE SAY THAT $\ell \in M$ IS A LOWER BOUND FOR A IS $\ell \leq a \quad \forall a \in A$

$M \equiv \mathbb{Q}$

3) $A = \{ r \in \mathbb{Q} \ ; \ 0 \leq r < 10 \}$

LET'S FIND AN UPPER BOUND $10 \in M$
$\in \mathbb{Q}$

$\forall a \in A \quad a \leq 10$

$1'000'000 \in M$

$\forall a \in A \quad a \leq 1'000'000$

→ UPPER BOUND MAY NOT BE UNIQUE

0 IS THE MINIMUM IN A ⇒ IT'S AUTOMATICALLY A LOWER BOUND

IF $\overline{m}$ IS A MAXIMUM ⇒ IT'S ALSO AN UPPER B.
$\underline{m}$ " MINIMUM ⇒ " " LOWER "

DEFINITION $M \neq \emptyset$, $(M, \leq)$ TOTALLY ORDERED

$A \subseteq M$ $A \neq \emptyset$. WE SAY THAT

$\bar{S} \in M$ IS THE SUPREMUM OF A

AND WE DENOTE IT WITH $\bar{S} = \sup A$ IF:

i) $a \leq \bar{S}$ $\forall a \in A$ [UPPER BOUND]

QUANTIFIER

ii) $\forall \varepsilon > 0$ $\exists \underline{b} \in \underline{A}$ SUCH THAT

$\equiv$ $\bar{S} - \varepsilon \leq b$ [SMALLEST OF THE UPPER BOUNDS]

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WE SAY THAT $\underline{S} \in M$ IS THE INFIMUM OF A

AND WE DENOTE IT WITH $\underline{S} = \inf A$ IF;

i) $\underline{S} \leq a$ $\forall a \in A$ [LOWER Bd]

ii) $\forall \varepsilon > 0$ $\exists b \in A$:

$\bar{S} + \varepsilon \geq b$ [LARGEST OF THE LOWER BOUNDS]

EXAMPLE, UPPER BOUND

$A = \{ r \in \underline{\mathbb{Q}} ; 0 \leq r < \underline{10} \}$

10 IS THE SMALLEST OF THE UPPER BOUNDS

i) 10 IS AN UPPER BOUND → OK

WE CLAIM $\bar{s} = 10$

FIX $\varepsilon = 0.1$

$10 - 0.1 = 9.9$ $\quad \underline{b = 9.91} = \frac{991}{1000} \in A$

FIX $\varepsilon = 0.01$ $\quad 10 - 0.01 = 9.99$

$b = 9.991 \quad \frac{9991}{10'000} \in A$

→ $b \in A ; \ \bar{s} - \varepsilon \leq b$

→ $\bar{s}$ IS THE SUPREMUM

PROPOSITION, IF SUP, inf EXIST, THEN THEY ARE UNIQUE.

DEFINITION • IF A HAS THE SUPREMUM THEN A IS "BOUNDED (FROM) ABOVE"

• IF A HAS THE INFIMUM, $\Rightarrow$ A IS "BOUNDED (FROM) BELOW"

• IF A HAS BOTH SUP & INF
A IS "BOUNDED"

EX 2) $A = \{0, 1, 2, 10\}$ BOUNDED

$\inf A = 0 = \min A$

$\sup A = 10 = \max A$

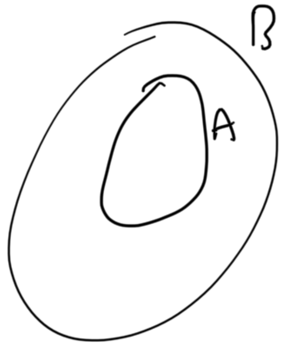
PROPERTIES

IF A HAS MAXIMUM $\max A = \sup A$

A " MINIMUM $\min A = \inf A$

PROPERTIES $(M, \leq)$ TOTALLY ORDERED

$M, A, B \neq \emptyset$ $A \subseteq B$



- $\sup A \leq \sup B$
- $\max A \leq \max B$
- $\inf A \geq \inf B$
- $\min A \geq \min B$

IF THESE QUANTITIES EXIST

$\max A = m_a \in A \qquad \max B = m_b \in B$

1) $a \leq m_a \quad \forall a \in A$

2) $b \leq m_b \quad \forall b \in B$

IN PARTICULAR WE CAN PLUG $b = m_a$ IN 2)

AND WE HAVE $m_a \leq m_b$.

DEFINITION $\Pi, A \neq \emptyset \quad (\Pi, \leq)$

- WE SAY THAT A IS "UNBOUNDED (FROM) ABOVE"

IF $\nexists$ UPPER BOUND FOR A

EX $A = \{ r \in \mathbb{Q} \quad r > 10 \}$

- WE SAY THAT A IS "UNBOUNDED (FROM) BELOW" IF $\nexists$ LOWER BOUND FOR A

$$A = \{ r \in \mathbb{Q} \; ; \; r < 10 \}$$

- WE SAY A IS "UNBOUNDED" IF A DOES NOT ADMIT UPPER NOR LOWER BOUND