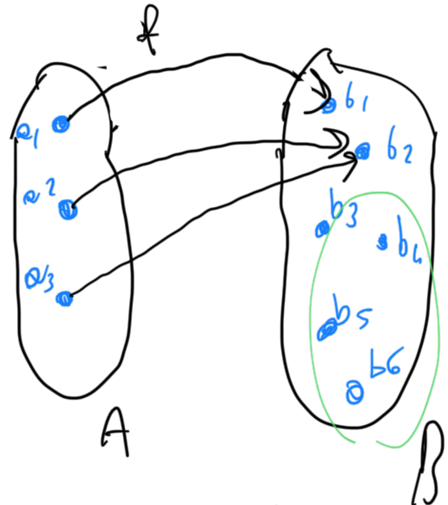
LECTURE 3, SEPTEMBER 10

(f, A, B)

$A \to$ DOMAIN

$B \to$ CODOMAIN



f ASSOCIATES EACH ELEMENT OF A TO A UNIQUE ELEMENT IN B

$b_1 = f(a_1)$

$b_2 = f(a_2)$

$b_2 = f(a_3)$

IN THIS EXAMPLE THERE IS NO POINT IN A MAPPED TO b_3, b_4, b_5, b_6

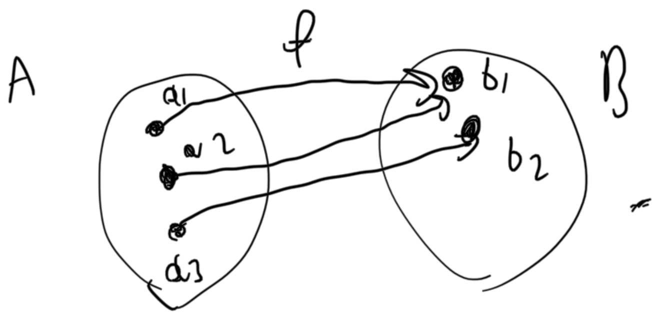
1) SURJECTIVITY

$f: A \to B$ IS "SURJECTIVE"

IF $\forall b \in B \ \exists a \in A : b = f(a)$

IN OTHER WORDS $f(A) = B$

IMAGE OF f = CO-DOMAIN



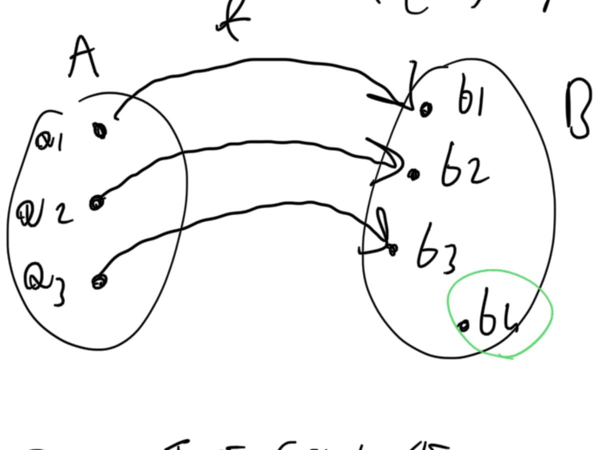
f IS SURJ.

2) INJECTIVITY

$f: A \to B$ IS INJECTIVE IF

$\forall a_1, a_2 \in A$ $a_1 \neq a_2$ WE HAVE

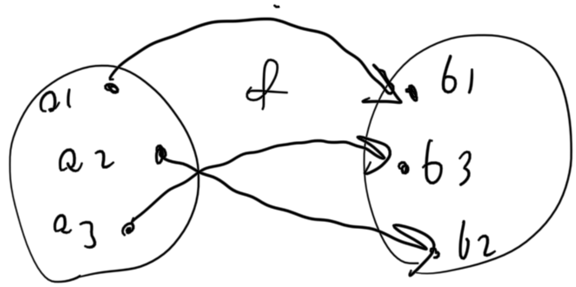
$$f(a_1) \neq f(a_2)$$



$f: A \to B$ INJECTIVE

3) BI-JECTIVE

$f: A \to B$ IS SURJ & INJ



$f(A) = B$

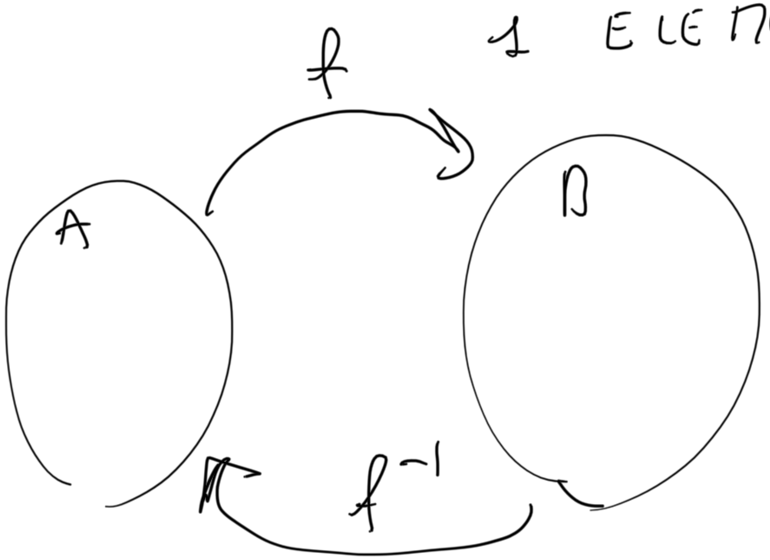
WHAT IS THE IDEA OF BIJECTIVITY

- $f(A) = B$ (SURJECTIVE)

  ALL THE POINTS IN THE CO-DOMAIN HAVE A COUNTER-IMAGE IN A

- $f^{-1}(b) = \{a\}$ = SINGLE/SINGLETON

  ↳ SET CONTAINING ONLY 1 ELEMENT



(f^{-1}, B, A) AND $f^{-1}: B \to A$ IS WELL DEFINED

IT'S A FUNCTION

BECAUSE IT MAPS EACH ELEMENT OF B TO A UNIQUE ELEMENT OF A

f^{-1} IS "WELL DEFINED" IF f IS BIJECTIVE.

<u>DEFINITION</u> let $f: A \to B$ BI-JECTIVE $\exists$ $f^{-1}: B \to A$ SUCH THAT f^{-1} ASSOCIATES EACH ELEMENT OF B TO A UNIQUE ELEMENT IN A

$$\underbrace{B \ni b}_{b \in B} \longmapsto f^{-1}(b) = a$$

WHERE a IS THE UNIQUE ELEMENT SUCH THAT $f(a) = b$

THIS $f^{-1}: B \to A$ IS CALLED THE "INVERSE" OF $f: A \to B$ $\square$

REMARK $f: A \to B$ BIJECTIVE IS ALSO CALLED ONE-TO-ONE OR INVERTIBLE

SYNONIMS

EXERCISE LET $f: A \to B$ BIJECTIVE THEN f^{-1} EXISTS AND f^{-1} IS BIJECTIVE

$\to$ DO IT AT HOME

f^{-1} IS SURJECTIVE AND INJECTIVE

EXAMPLES

$\boxed{1}$ $f: \mathbb{R} \to \mathbb{R}$ $x \to f(x) = x^2$

DOMAIN, CO-DOMAIN, IMAGE OF f,

IS IT SURJ., IS IT INJ., BI-J ?

IF SO, COMPUTE f^{-1}

- DOMAIN $= \mathbb{R}$, CO-DOMAIN $= \mathbb{R}$
- IMAGE OF f ? $\left((\cdot)^2, \mathbb{R}, \mathbb{R} \right)$

(f, A, B)

$$f(\mathbb{R}) = \overbrace{\{x \in \mathbb{R} \; ; \; x \geq 0\}}^{\mathbb{R}^+} \subseteq \mathbb{R}$$

- $\Rightarrow f$ IS NOT SURJECTIVE

$$\underbrace{f(\mathbb{R})}_{\text{IMAGE}} \neq \underbrace{\mathbb{R}}_{\text{CO-DOMAIN}}$$

- IS f INJECTIVE ?

$$+1, -1 \xrightarrow{f} +1$$

IT'S NOT INJECTIVE, $\Rightarrow$ IT'S NOT BI-J

$\rightarrow \nexists \; f^{-1} : \mathbb{R} \to \mathbb{R}$

- COMPUTE COUNTER-IMAGE OF f

$$\mathbb{R} \xrightarrow{f} \mathbb{R}$$

$$f^{-1}(y) = \begin{cases} \emptyset \quad \text{ZERO SET} & y < 0 \\ 0 \quad \text{ZERO NUMBER} & y = 0 \\ \{\pm\sqrt{y}\} & y > 0 \end{cases}$$

$y \in \mathbb{R}$ CO-DOMAIN

$\boxed{2}$ $f : \mathbb{R} \to \mathbb{R}^+$ $\quad x \to x^2$

$\mathbb{R}^+ := \{x \in \mathbb{R} \; ; \; x \geq 0\}$

- DOMAIN : $\mathbb{R}$, CO-DOMAIN IS $\mathbb{R}^+$
- IS IT SURJECTIVE ? → YES

CO-DOMAIN COINCIDES WITH IMAGE OF f

IMAGE OF f

- $f(\mathbb{R}) = \underline{\mathbb{R}^+} = \{x \in \mathbb{R} \; ; \; x \geq 0\}$ ✓

$\left((\cdot)^2, \mathbb{R}, \mathbb{R}^+\right) \neq \left((\cdot)^2, \mathbb{R}, \mathbb{R}\right)$

DIFFERENT FUNCTIONS !!!

- IS $f : \mathbb{R} \to \mathbb{R}^+$ INJECTIVE ? NO

BECAUSE $+1, -1 \to +1$

$$f^{-1}(y) \begin{cases} 0 & y = 0 \\ \{\pm\sqrt{y}\} & y > 0 \end{cases}$$

THIS FUNCTION IS NOT BI-JECTIVE

$\boxed{3}$ $f: \mathbb{R}^+ \to \mathbb{R}^+$ $\quad f: x \to x^2$

DOMAIN CO-DOMAIN

- IMAGE OF f $\quad f(\mathbb{R}^+) = \underline{\mathbb{R}^+}$

CO-DOM. COINCIDES WITH IMAGE

$\to$ SURJECTIVE

- IS IT INJECTIVE ?

$x_1 \in \mathbb{R}^+, \; x_2 \in \mathbb{R}^+ \quad x_1 \neq x_2$

$\Rightarrow$ WE HAVE $\; x_1^2 \neq x_2^2$

$\to$ IT IS INJECTIVE

$\pm 1 \to 1$

THIS IS SO BECAUSE NEGATIVE NUMBERS DO NOT BELONG TO OUR DOMAIN

- f IS BI-JECTIVE
- $\exists \; f^{-1}: \mathbb{R}^+ \to \mathbb{R}^+$

$$f^{-1}(y) = \{+\sqrt{y}\} \quad \forall y \in \mathbb{R}^+$$

$$(f^{-1}, \mathbb{R}^+, \mathbb{R}^+) = (+\sqrt{\ }, \mathbb{R}^+, \mathbb{R}^+)$$

WE ARE SEEING ROLE OF DOMAIN AND CO-DOMAIN IN FUNCTIONS

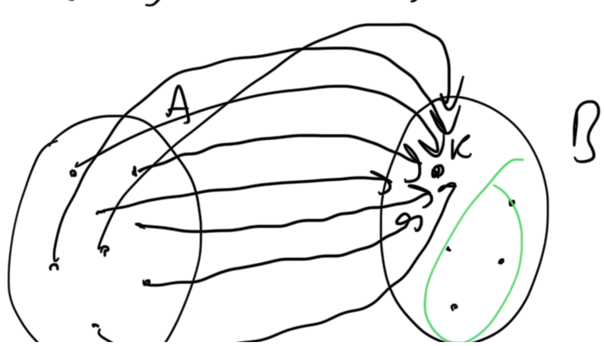
$$((\cdot)^2, \mathbb{R}, \mathbb{R}) \neq ((\cdot)^2, \mathbb{R}, \mathbb{R}^+) \neq ((\cdot)^2, \mathbb{R}^+, \mathbb{R}$$

THE CONSTANT FUNCTION

$k \in \mathbb{R}$ LET'S FIX $k \in \mathbb{R}$

$(k, \mathbb{R}, \mathbb{R})$

(k, A, B) $A, B \subseteq \mathbb{R}$ $k \in B$



- THE CONSTANT FUNCTION $(k, \mathbb{R}, \mathbb{R})$ IS <u>NOT</u> SURJECTIVE
- $(k, \mathbb{R}, \mathbb{R})$ IS <u>NOT</u> INJECTIVE

THE IDENTITY $\qquad A \subseteq B$

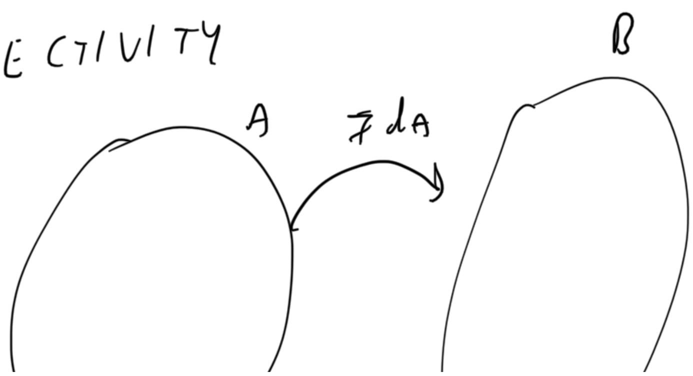
$(\underline{Id_A}, A, B) \qquad Id_A : A \to B$

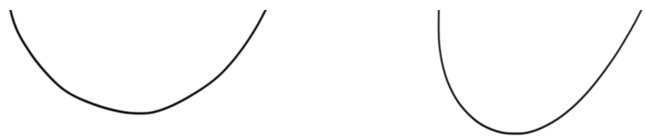
$$A \ni a \to Id_A(a) = a$$

- INJECTIVITY $\quad$ IF $\underset{\displaystyle A}{\overset{\displaystyle a_1}{\cap}} \neq \underset{\displaystyle A}{\overset{\displaystyle a_2}{\cap}} \Rightarrow$

$$\underset{\displaystyle a_1}{\overset{\displaystyle Id_A(a_1)}{\Vert}} \neq \underset{\displaystyle a_2}{\overset{\displaystyle Id_A(a_2)}{\Vert}}$$

- SURJECTIVITY





WE NEED TO COMPARE IMAGE AND CO-DOMAIN

$Id_A(A) = A$ CO-DOMAIN IS B

IF $A = B \rightarrow Id_A$ IS SURJECTIVE

IF $A \subset B \rightarrow$ " " NOT SURJ

STRICTLY CONTAINED

$A \subseteq B$

$$(Id_A)^{-1}(y) = \begin{cases} \emptyset & \text{IF } y \notin A \\ \{y\} & \text{IF } y \in A \end{cases}$$

COUNTER-IMAGE

IF $A = B$

$(Id_A, A, A) \rightarrow$ INJ + SURJ $\Rightarrow$ BI-JECTIVE

SOME MORE FUNCTIONS / OPERATIONS

BETWEEN FUNCTIONS -

$f : A \to B$, $g : A \to B$

$f + g$? $(f+g) : A \to B$

$$\underbrace{(f+g)}_{\text{NEW FUNCTION}}(a) := f(a) + g(a) \quad \forall a \in A$$

$(f+g, A, B)$ OBTAINED FROM

(f, A, B) AND (g, A, B)

— " —

$$(f-g)(a) := f(a) - g(a)$$

$(f-g, A, B)$

— " —

$$(f \cdot g)(a) := f(a) \cdot g(a)$$

$(f \cdot g, A, B)$

— " —

$$(f/g)(a) := f(a)/g(a)$$

PROVIDED $g(a) \neq 0$

$(f/g, A, B)$

COMPOSITION OF 2 FUNCTIONS

$f; A \to B$, $g; B \to C$

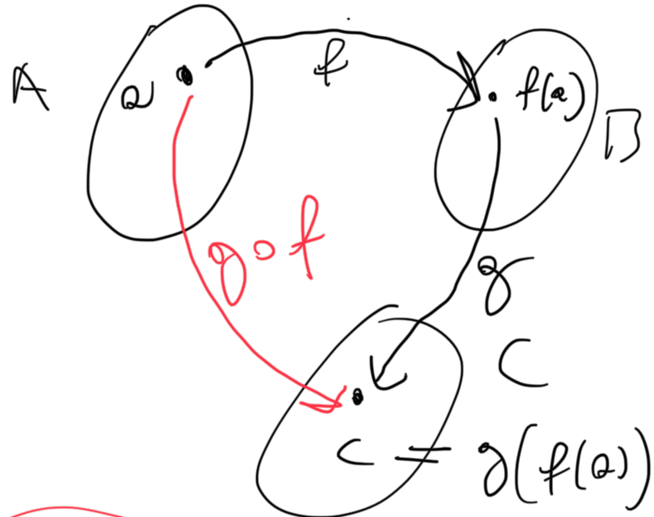
I DEFINE THE COMPOSITION OF f WITH g THE NEW FUNCTION

$$g \circ f : A \to C$$

THAT ASSOCIATES TO EACH $a \in A$ THE ELEMENT $c = g(f(a))$ –

$g \circ f ; A \to C$ DEFINED

$$A \ni a \longrightarrow c = (g \circ f)(a) = g(f(a)) \in C$$



REMARK $g \circ f$ = "g composed with f"

$$g \circ f \neq f \circ g \leftarrow$$

PROPERTIES

- IF $f: A \to B$ & $g: B \to C$ ARE INJECTIVE, THE $g \circ f : A \to C$ IS INJECTIVE
- IF $f: A \to B$ & $g: B \to C$ ARE SURJECTIVE $\Rightarrow$ $g \circ f : A \to C$ IS SURJECTIVE
- IF $f: A \to B$, $g: B \to C$ ARE BI-JECTIVE $\Rightarrow$ $g \circ f : A \to C$ IS BI-JECTIVE

PROOF → TRY TO DO YOURSELF.

<u>PROPOSITION</u> THE RULE TO COMPUTE THE INVERSE OF A COMPOSITION

$$(g \circ f) : A \to C$$

WITH $f: A \to B$ BIJECTIVE

$g: B \to C$ "

$$\Rightarrow (g \circ f)^{-1} : C \to A$$

$$\boxed{(g \circ f)^{-1}(c) = a = (f^{-1} \circ g^{-1})(c)}$$

$\forall c \in C$

REMARK

LET $h: A \to B$ WHICH IS BIJECTIVE

$h^{-1}: B \to A$

$h: A \to B$ $\quad a \quad h(a) = b$

$h^{-1}: B \to A$ $\quad b \quad h^{-1}(b) = a$

$$h^{-1} \circ h : A \to A$$

$$h^{-1} \circ h = Id_A$$

PROOF WE START $\left(\overbrace{g\circ f}^{h}\right)^{-1}\circ\left(\overbrace{g\circ f}^{h}\right) = Id_A$

Q = I LET'S CONTINUE MONDAY

$\boxed{\text{I WILL POST EXERCISES LATER TODAY ON THE WEBSITE}}$