LECTURE 2, SEPTEMBER 8

APOSTOL → COVERS ALL TOPICS
INCLUDED SET THEORY

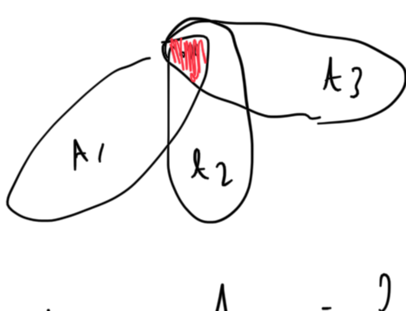
ADAMS / STEWART → EXCELLENT FOR DIFFERENTIAL
CALCULUS / INTEGRATION

IF YOU HAVE OTHER BOOKS ALREADY
I CAN CHECK IF THEY ARE OK FOR THIS COURSE

REVIEW OF BASIC SET THEORY

$$A_1 \cap A_2 = \{x \in A_1 \wedge x \in A_2\}$$

$$A_1 \cap A_2 \cap A_3 = \{x \in A_1 \wedge x \in A_2 \wedge x \in A_3\}$$

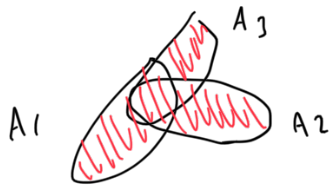


$$A_1 \cap A_2 \ldots A_n = \{x \in A_1 \wedge A_2 \ldots \wedge A_n\}$$

$n \in \mathbb{N}$

$A_1 \cup A_2 \dots A_n = \{x \in A_1 \vee x \in A_2 \dots x \in A_n\}$

$n \in \mathbb{N}$



CARDINALITY OF A

$\# A$ OR $|A|$ = NUMBER OF ELEMENTS OF A

$\# A \in \mathbb{N}$ WE SAY THAT CARDINALITY OF A IS FINITE OR A IS A FINITE SET

$A = \{$STATES OF THE U.S.A.$\}$

$\# A = 50 \in \mathbb{N}$ A IS A FINITE SET

$A = \mathbb{N} := \{$NATURAL NUMBERS$\} = \{0, 1, 2, 3, \dots\}$

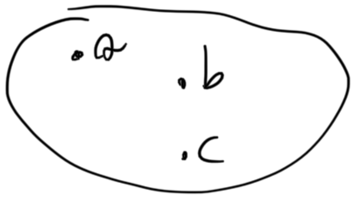
IS NOT FINITE

$\# A = \infty$ INFINITE

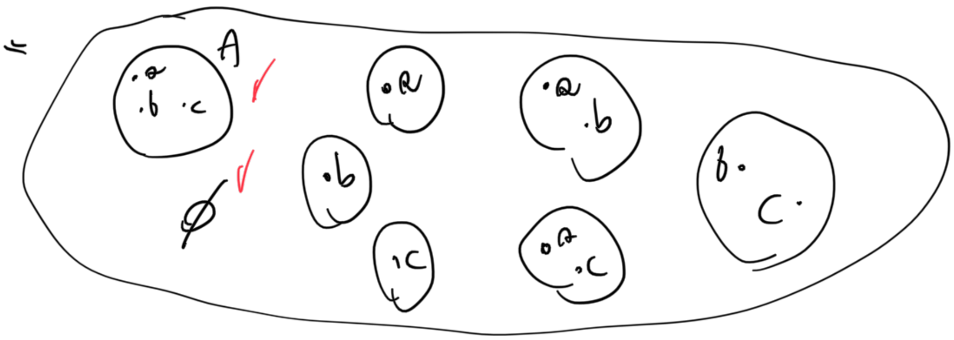
LET A BE A SET

$P(A)$ = SET OF ALL THE SUBSETS OF A

"THE SET OF THE PARTS OF A"

$A =$ 

$A = \{a, b, c\}$

$P(A) =$ 

$\#A = 3$ IF $\#A = n \in \mathbb{N}$

$$\#(P(A)) = 2^n$$

$\#A = 3$ $\quad \# P(A) = 2^3 = 8$

CARDINALITY OF $P(A)$ "2^A"

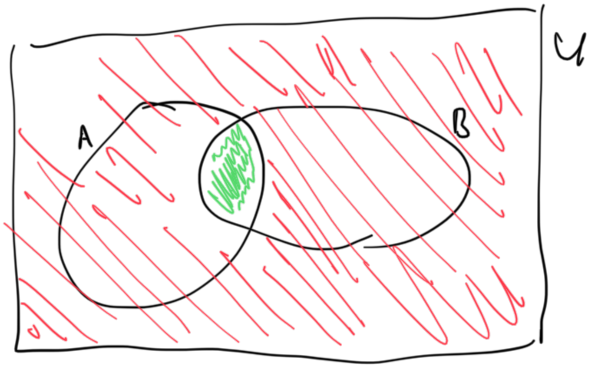
DE MORGAN'S LAWS

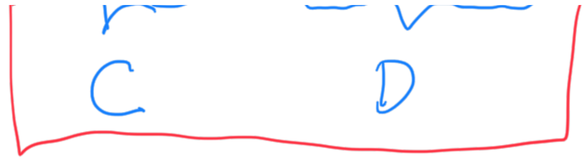
- $(A \cup B)^c = A^c \cap B^c$ (**)
- $(A \cap B)^c = A^c \cup B^c$ (*)

(**) DO IT AS AN EXERCISE.

PROOF OF (*)

$$\boxed{(A \cap B)^c = A^c \cup B^c}$$





A WAY TO SHOW $C = D$

$$\left.\begin{array}{l} C \subseteq D \\ C \supseteq D \end{array}\right. \Rightarrow C = D$$

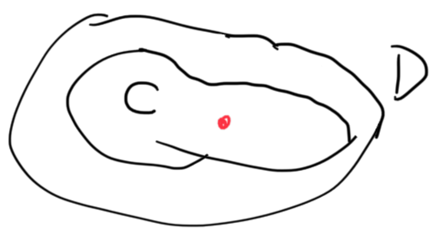
OUR STRATEGY IS TO SHOW I) $C \subseteq D$

II) $D \subseteq C$

WE START WITH I).

WHAT DOES IT MEAN THAT $C \subseteq D$?

IT MEANS THAT A GENERAL ELEMENT OF C IS ALSO CONTAINED IN D.



$$C := (A \cap B)^c$$

$$D := A^c \cup B^c$$

$$x \in C = (A \cap B)^c$$

$$x \notin (A \cap B)$$

$$x \notin A \lor x \notin B$$

$x \in A^c \vee x \in B \Rightarrow x \in A^c \cup B^c = D$

$\rightarrow$ AS REQUIRED, STEP I.

STEP II.) WE WANT TO SHOW $D \subseteq C$

$$A^c \cup B^c \subseteq (A \cap B)^c \quad (*)$$

$d \in A^c \cup B^c$ WE USE "CONTRADICTION ARGUMENT"

OR "ABSURDUM - METHOD"

- WE DENY THE THESIS AND WE SHOW THIS LEADS TO A CONTRADICTION
  $\rightarrow$ THE THESIS IS IN FACT TRUE

WE DENY $(*)$ WE ASSUME $A^c \cup B^c \not\subseteq (A \cap B)^c$

$d \in A^c \cup B^c$ $\qquad d \notin (A \cap B)^c$

$d \in (A \cap B)$

$d \in A \wedge d \in B$.

$d \notin A^c \vee d \notin B^c$.

$d \notin A^c \cup B^c$

CONTRADICTION!

→ THE THESIS CANNOT BE DENIED

$$A^c \cup B^c \subseteq (A \cap B)^c \qquad \square$$

REMARK THIS PROOF ABOVE WORKS IF $A \cap B \neq \emptyset$

WHAT HAPPENS IF $A \cap B = \emptyset$?

IN STEP I) WE SHOWED THAT

$$(A \cap B)^c \subseteq A^c \cup B^c \subseteq U$$

$$\overset{\parallel}{U} \qquad A^c \cup B^c = U \quad \text{IF } A \cap B = \emptyset$$

RELATIONS

GIVEN $A \neq \emptyset$ WE DEFINE A RELATION $\mathcal{R}$ ON A A SUBSET OF THE CARTESIAN PRODUCT $A \times A$.

GIVEN $\mathcal{R}$ WE SAY THAT x IS IN RELATION WITH y IF $(x, y) \in \mathcal{R}$ AND WE INDICATE

$x \mathcal{R} y$

- EQUIVALENCE
- ORDER

EQUIVALENCE RELATION

$xRy \qquad x \sim y$ IF THE FOLLOWING HOLDS:

-) $x \sim x \quad \forall x \in A$ (REFLEXIVE PROPERTY)

-) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in A$ (SYMMETRIC P.)

-) $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$ (TRANSITIVE P.)
$\forall x, y, z \in A$

$xRy \quad x \sim y$

EXAMPLE $=$ EQUALITY IN $\mathbb{N}$

$A =$ FIRST YEAR STUDENTS AT N.Y. UNIVERSITY

$\sim$ = TO BE BORN IN THE SAME STATE

DEFINITION "EQUIVALENCE CLASS"

GIVEN $A \neq \emptyset$, AND AN $\sim$ ON A

GIVEN $x \in A$, I DEFINE

$$[x] := \{ y \in A : y \sim x \}$$

$\parallel$

EQUIVALENCE CLASS OF x

x = ANNA (BORN IN CALIFORNIA)

[ANNA] = {ALL FIRST YEAR STUDENTS N.Y. BORN IN CALIFORNIA}

BOB ALSO BORN IN CALIFORNIA

[ANNA] = [BOB]

INSTEAD CHARLIE BORN IN ALASKA

[CHARLIE] $\neq$ [ANNA]

$$x \not\sim y \Rightarrow [x] \neq [y]$$

$$x \sim y \Rightarrow [x] = [y]$$

DEFINITION_ "QUOTIENT SET"

$$Q := A/\sim = \{[x], x \in A\}$$

IT'S EASY TO CONSTRUCT QUOTENT SET FOR

FIRST YEAR STUDENTS OF N.Y.U. FOR THE

EQUIVALENCE RELATION "TO BE BORN IN THE SAME STATE"

$Q = \{ [ANNA], [CHARLIE], \dots \}$

$\#Q = n \in \mathbb{N} \qquad n \le 50$

ORDER RELATION (PARTIAL ORDER RELATION)

GIVEN $A \neq \emptyset$ IF THE FOLLOWING HOLD

-) REFLEXIVE $\quad xRx \quad \forall x \in A$

-) $(xRy \wedge yRx) \Rightarrow x = y \quad \forall x,y \in A$ [ANTI-SYMMETRIC)

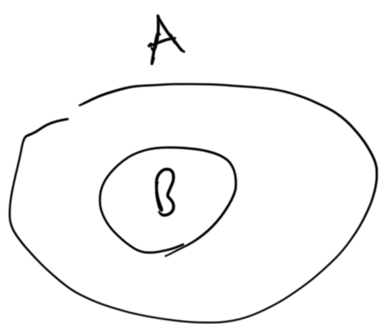
-) TRANSITIVE $\quad xRy \wedge yRz \Rightarrow xRz$

$\forall x, y, z,$

$x \precsim y$ AN EXAMPLE IS $\boxed{\le}$ ←

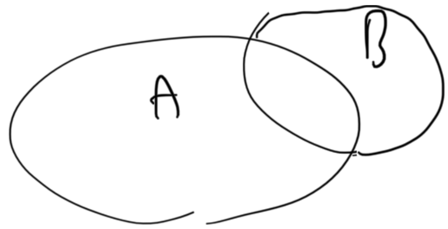
$x \le y \quad y \le x \Rightarrow x = y$

$x \le y \quad y \le z \Rightarrow x \le z$



$B \boxed{\subseteq} A$

WE SAY THAT $\le$ IS A "TOTAL" ORDER RELATION ON A SET A



FUNCTIONS

WE START WITH BEING VERY FORMAL

A FUNCTION IS A TRIPLE (f, A, B)

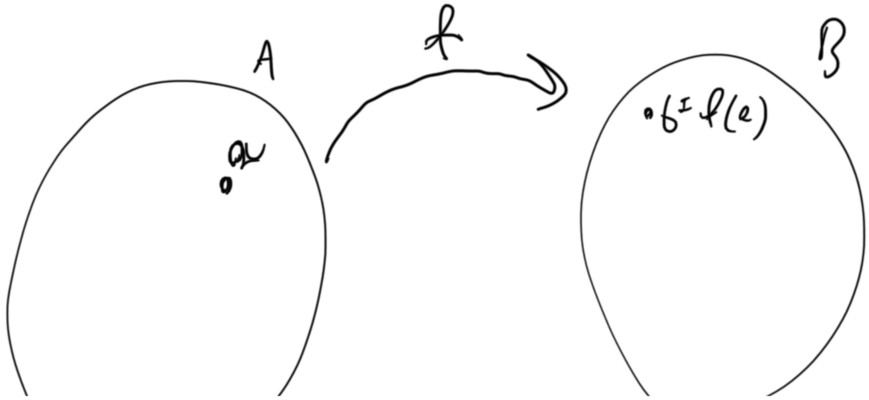
A, B A = "DOMAIN", B = "CO-DOMAIN"

AND f IS A LAW THAT ASSOCIATES TO EACH ELEMENT OF A A UNIQUE ELEMENT IN B

$$\forall a \in A \quad \exists!\, b \in B : b = f(a)$$

$$f : A \to B \qquad a \mapsto b = f(a)$$ = "IMAGE" OF a

INDICATES ASSOCIATION BETWEEN a, b VIA THE FUNCTION f.



$\boxed{a}$ $I \subseteq A$ DOMAIN

$f(I)$ = SUBSET OF B WHICH CONTAINS THE IMAGES OF ALL ELEMENTS IN I

$$f(I) := \{ b \in B : b = f(a), a \in I \} \subseteq B$$

$\boxed{a}$ $I \subseteq A$ DOMAIN

$f(I)$ = SUBSET OF B WHICH CONTAINS THE IMAGES OF ALL ELEMENTS IN I

$$f(I) := \{ b \in B : b = f(a), a \in I \} \subseteq B$$

IF $a \in I$

" f RESTRICTED TO I "

$I \subseteq A$

RESTRICTION OF f ON THE SET I

$$\left(f|_I, I, B\right)$$

FROM A FORMAL POINT OF VIEW $(f, A, B) \neq (f|_I, I, B)$

- DEFINITION – REPLACE A WITH I IN $\boxed{a}$

$$f(A) := \{ b \in B ; b = f(a) \ \forall a \in A \} \subseteq B$$

IMAGE OF THE DOMAIN = ALSO CALLED "THE IMAGE OF f"

• $\boxed{c}$ COUNTER-IMAGE

GIVEN $b = f(a)$ THE IMAGE OF a VIA f.

$b \in B$

WE CALL "THE COUNTER-IMAGE" OF b, THE SET

$$f^{-1}(b) := \{a \in A ; f(a) = b\} \subseteq A$$

$$f^{-1}(J) := \{a \in A ; f(a) \in J\} \subseteq A$$

$J \subseteq B$ A SUBSET OF THE CO-DOMAIN

FINER PROPERTIES, SURJECTIVITY
INJECTIVITY
BI-JECTIVITY

$\hookrightarrow$ THIS ON WEBSITE LATER TODAY