Exercises, September 10, 2021

Basic properties of functions

Exercise 1

Let $\mathbb{R}^+ := \{x \in \mathbb{R}, \text{such that } x \geq 0\}$, $\mathbb{R}^- := \{x \in \mathbb{R}, \text{such that } x \leq 0\}$. Consider the following functions, where $x \to f_1(x) = x^2$, $x \to f_2(x) = x^3$, $x \to f_3(x) = x^4$

a) $(f_1, \mathbb{R}^-, \mathbb{R}^+)$

b) $(f_2, \mathbb{R}, \mathbb{R})$

c) $(f_2, \mathbb{R}^+, \mathbb{R}^+)$

d) $(f_3, \mathbb{R}, \mathbb{R})$

e) $(f_3, \mathbb{R}, \mathbb{R}^+)$

f) $(f_3, \mathbb{R}^+, \mathbb{R}^+)$

For each of these functions discuss domain, co-domain, image, surjectivity, injectivity, bijectivity, counterimage and, if possible, find the inverse function.

Exercise 2

Consider the function $(f, A, \mathbb{R})$ where

$$f(x) := \frac{ax + b}{cx + d}. \tag{1.12}$$

For $a, b, c, d \in \mathbb{R}$ find the largest domain A where it makes sense to define $(f, A, \mathbb{R})$ and then discuss for which values of $a, b, c, d \in \mathbb{R}$ the function $(f, A, \mathbb{R})$ is injective.

Hint. To determine A you have to discuss $cx + d = 0$ for $c, d \in \mathbb{R}$.

Exercise 3

Consider the sets

$$A := \{\text{students of Shanghai University}\} \tag{1.13}$$

and

$$B := \{\text{shoe sizes}\} \tag{1.14}$$

and consider the map f that assigns to each student their own shoe size.

- Is the function (f, A, B) injective? Explain why.

Then, consider the set

$$C := \{\text{Shanghai University students' ID numbers}\} \tag{1.15}$$

and the function g that associates to each student their own University ID number.

- Is the function (h, A, C) injective? Explain why.
- Is the function (h, A, C) surjective? Explain why.
- Is the function (h, A, C) bijective? Explain why.

Sketch of the solutions

Exercise 1

a) Injective, surjective, bijective, $f_1^{-1} = -\sqrt{\cdot}$, the counterimage $f_1^{-1}(\mathbb{R}^+) = \mathbb{R}^-$.

b) Injective, surjective, bijective, $f_2^{-1} = \sqrt[3]{\cdot}$, the counterimage $f_2^{-1}(\mathbb{R}) = \mathbb{R}$.

c) Injective, surjective, bijective, $f_2^{-1} = \sqrt[3]{\cdot}$, the counterimage $f_2^{-1}(\mathbb{R}^+) = \mathbb{R}^+$.

d) not injective, not surjective, the counterimage $f_3^{-1}(y) = \{\pm\sqrt[4]{y}\}$ if $y > 0$, $f_3^{-1}(0) = \{0\}$ and $f_3^{-1}(y) = \emptyset$ if $y < 0$.

e) Not injective, surjective, the counterimage $f_3^{-1}(y) = \{\pm\sqrt[4]{y}\}$ if $y > 0$ and $f_3^{-1}(0) = \{0\}$.

f) Injective, surjective, bijective, $f_3^{-1} = \sqrt[4]{\cdot}$, the counterimage $f_3^{-1}(\mathbb{R}^+) = \mathbb{R}^+$.

Exercise 2

First, if $c = d = 0$ the fraction is not defined. We continue assuming $(c, d) \neq (0, 0)$.

Then, suppose $c = 0, d \in \mathbb{R}, d \neq 0$. In this case $A = \mathbb{R}$. In the general case $c \neq 0$ we have $A = \{x \in \mathbb{R} : cx + d \neq 0\}$ and $f : A \to \mathbb{R}$ is injective provided $ad \neq cb$.

Exercise 3

Intuitively, $f : A \to B$ cannot be injective as it is very easy to find counterexamples. $h : A \to C$ is injective, surjective and bijective because each student has their own individual IDs and duplicate IDs are not allowed.